

Inverse Scattering at a Fixed Energy for Long-Range Potentials ^{*†}

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Abstract

In this paper we consider the inverse scattering problem at a fixed energy for the Schrödinger equation with a long-range potential in $\mathbb{R}^d, d \geq 3$. We prove that the long-range part can be uniquely reconstructed from the leading forward singularity of the scattering amplitude at some positive energy.

1 Introduction

Our goal is to study the inverse scattering problem at a fixed energy for the Schrödinger equation with a long-range potential. We give a method for the unique reconstruction of the long-range part of the potential at infinity. The reconstruction of the short-range part is similar to the procedure suggested in [13] where the same problem for short-range potentials

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was treated. To a certain extent, this paper can be considered as continuation of [13], but we do not dwell here upon reconstruction of the short-range part of a long-range potential.

For a short survey of different formulations of the inverse scattering problem see [13]. Here we just mention the contributions to long-range inverse scattering. Isozaki and Kitada [7] consider potentials that satisfy

$$|\partial^\alpha V(x)| \leq C_\alpha (1 + |x|)^{-\rho - |\alpha|}, \quad \rho > 1/2, \quad \text{for all } \alpha. \quad (1.1)$$

Using stationary methods, they proved that the high-energy limit $\lambda \rightarrow \infty$ of the scattering matrix $S(\lambda)$ determines uniquely a potential and give a method for its reconstruction. A similar result was obtained by Yafaev [18] for an arbitrary $\rho > 0$. More precisely, the Fourier transform of the potential is reconstructed in [7], [18] by taking the high-energy limit of the scattering amplitude with fixed momentum transfer. Enss and Weder [4] proved a similar high-energy uniqueness result and gave a reconstruction method using a time-dependent approach. They consider a large class of long-range potentials that are allowed to have singularities as well as N-body potentials. In [4] the X-ray transforms of the potentials are uniquely reconstructed from the action of the scattering operator on appropriate high-energy states.

As mentioned above, we are interested in this paper in the inverse scattering problem at a fixed energy in the case of long-range potentials in \mathbb{R}^d , $d \geq 3$. Surprisingly, this problem has received little attention. In [8] it was proven that the scattering matrix at a fixed energy uniquely determines the asymptotics of a short-range potential on the background of a Coulomb potential, what actually is a different problem. We also mention paper [9], where using the method of [4] the asymptotics of a long-range potential is uniquely reconstructed if the scattering matrix is known on some (perhaps arbitrarily small) energy interval.

We suppose that $V \in C^\infty(\mathbb{R}^d)$, $d \geq 3$, and that, for sufficiently large $|x|$,

$$V(x) = \sum_{j=1}^N V_j(x) + V_{sr}(x), \quad (1.2)$$

where $V_j \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a homogeneous function of order $-\rho_j$, i.e., $V_j(tx) = t^{-\rho_j} V_j(x)$ for all $t > 0$, with $1/2 < \rho_1 < \rho_2 < \dots < \rho_N \leq 1$, and $V_{sr} \in C^\infty(\mathbb{R}^d)$ is a short-range potential that satisfies (1.1) for some $\rho_{sr} > 1$. Our objective is to uniquely reconstruct

all functions $V_j(x)$, $j = 1, \dots, N$, if only the leading forward singularity of the scattering amplitude is known at some fixed energy $\lambda > 0$. Of course, we do not aim to reconstruct the whole potential, as is the case when the high-energy limit of the scattering matrix is known. The key issue here is that the forward singularity of the scattering amplitude contains all the information about the behaviour of the potential at infinity, and this allows us to uniquely reconstruct its asymptotic expansion at infinity. In particular, the leading singularity is sufficient for reconstruction of all long-range terms. Technically, we follow sufficiently closely our previous paper [13], where short-range potentials that satisfy (1.1) with $\rho > 1$ were considered.

Under condition (1.1) the scattering matrix $S(\lambda)$, where $\lambda > 0$ is the energy of a quantum particle, is a unitary operator on $L^2(\mathbb{S}^{d-1})$. Formally, the scattering matrix can be considered as an integral operator, that is

$$(S(\lambda)f)(\nu) = \int_{\mathbb{S}^{d-1}} s(\nu, \omega; \lambda) f(\omega) d\omega,$$

with integral kernel (the scattering amplitude) $s(\nu, \omega; \lambda)$. Here ω is the direction of the incident beam of particles and ν is the direction of observation. We emphasize that our definition of the scattering amplitude is somewhat different from the short-range case where the integral kernel of $S(\lambda)$ and the scattering amplitude differ by the Dirac delta-function. In the long-range case the delta-function disappears from the integral kernel. As is well known [1], the scattering amplitude is C^∞ -function away from the diagonal $\nu = \omega$, but its diagonal singularity is very wild [15, 18].

Actually, it is more convenient (especially, in the long-range case) to consider the scattering matrix as a pseudodifferential operator. It means that

$$(S(\lambda)f)(\nu) = (2\pi)^{-d+1} k^{d-1} \int_{\Pi_\omega} \int_{\mathbb{S}^{d-1}} e^{-ik\langle y, \nu \rangle} a(y, \omega; \lambda) f(\omega) dy d\omega, \quad k = \lambda^{1/2}, \quad (1.3)$$

where Π_ω is the hyperplane in \mathbb{R}^d orthogonal to ω , $y \in \Pi_\omega$ is known in the physics literature as the impact parameter and $a(y, \omega; \lambda)$ is the right symbol of the pseudodifferential operator $S(\lambda)$. It is related to the scattering amplitude by the formula

$$s(\nu, \omega; \lambda) = (2\pi)^{-d+1} k^{d-1} \int_{\Pi_\omega} e^{-ik\langle y, \nu \rangle} a(y, \omega; \lambda) dy. \quad (1.4)$$

Here and below we use freely the terminology of pseudodifferential calculus (see, e.g., [11], [12]). For example, expressions such as (1.3) or (1.4) are understood as oscillating integrals. Note that our definitions differ from the standard ones by the factor $-k$ in the phase in (1.3) or (1.4). The fact that the scattering matrix is well defined as a pseudodifferential operator was proven in [18]. In particular, it was shown there that its symbol $a(y, \omega; \lambda)$ belongs to the Hörmander class $\mathcal{S}_{\rho, 1-\rho}^0 = \mathcal{S}_{\rho, 1-\rho}^0(T^*\mathbb{S}^{d-1})$ if $\rho < 1$ and to the class $\mathcal{S}_{1-\varepsilon, \varepsilon}^0$ for any $\varepsilon > 0$ if $\rho = 1$. Here $T^*\mathbb{S}^{d-1}$ is the cotangent bundle of the unit sphere, that is the set of points (y, ω) such that $\omega \in \mathbb{S}^{d-1}$ and $y \in \Pi_\omega$. The class $\mathcal{S}_{\rho, 1-\rho}^0$ fits in the standard pseudodifferential calculus exactly in the case $\rho > 1/2$.

We proceed from the results of [15, 18] where it was shown that the principal symbol $a_0(y, \omega; \lambda)$ of $S(\lambda)$ is given by the equation

$$a_0(y, \omega; \lambda) = e^{-i(2k)^{-1}\Phi(y, \omega)} \quad (1.5)$$

where

$$\Phi(y, \omega) = \Phi(y, \omega; V) = \int_{-\infty}^{\infty} (V(y + t\omega) - V(t\omega)) dt. \quad (1.6)$$

Roughly speaking, our approach consists of the following steps.

1. Given $S(\lambda)$, we find its principal symbol $a_0(y, \omega; \lambda)$. Actually, it suffices for us to know the operators

$$S_{\omega_0}(\lambda) = \varphi_{\omega_0} S(\lambda) \varphi_{\omega_0}, \quad (1.7)$$

where $\omega_0 \in \mathbb{S}^{d-1}$ is an arbitrary point and φ_{ω_0} is multiplication by the function $\varphi_{\omega_0} \in C^\infty(\mathbb{S}^{d-1})$ such that $\varphi_{\omega_0}(\omega) = 1$ in some, arbitrary small, neighborhood O_{ω_0} of the point ω_0 . Then we use the fact that $a_0(y, \omega; \lambda)$ coincides with the principal symbol of the pseudodifferential operator $S_{\omega_0}(\lambda)$ for $\omega \in O_{\omega_0}$.

2. For the reconstruction of the long-range part of V , it suffices to know function (1.6). Clearly, if V is asymptotically homogeneous of order $-\rho$, then Φ is an asymptotically homogeneous function of order $-\rho + 1$ of the variable y (except the case $\rho = 1$ when Φ has a logarithmic behavior at infinity). Under assumption (1.2) the contributions to Φ of different functions V_j can clearly be separated in (1.6). Then we can directly reconstruct V_j by the inversion of the Radon transform (in some two-dimensional plane not passing through the origin).

Eventually, our method extends to potentials that satisfy (1.1) with $\rho > 0$. However, in the general case two new additional difficulties should be taken into account. The first is that the phase function in (1.5) is given by a more complicated formula than (1.6) although (1.6) remains its first approximation (see [10], [18]). The second difficulty is that for $\rho \leq 1/2$ the symbol of the scattering matrix is oscillating too rapidly so that the standard pseudodifferential operator calculus cannot be applied. In this case one has to use more specific results for pseudodifferential operators with oscillating symbols [16].

The paper is organized as follows. In Section 2, we recall different definitions of the wave operators in the long-range case. The scattering operator and the scattering matrix are also introduced there. Following [15, 18], we give in Section 3 the description of leading forward singularity of the scattering amplitude. The classical inversion formula for the Radon transform is recalled in Section 4. In Section 5 we uniquely reconstruct the long-range part of the potential.

2 Long-Range Scattering

Here we recall some basic definitions of long-range scattering theory; see, e.g., [17], for more details. We consider the Schrödinger operator

$$H = -\Delta + V(x)$$

with potential $V(x)$ in the space $L^2(\mathbb{R}^d)$ where $d \geq 2$. If V is a real and bounded function, then the Hamiltonian H is well defined on the Sobolev class $H^2(\mathbb{R}^d)$ and is self-adjoint in the space $L^2(\mathbb{R}^d)$. Let us denote by $H_0 = -\Delta$ the “free” Hamiltonian corresponding to the case $V = 0$. Under assumption (1.1) the operator H has no singular continuous spectrum, its absolutely continuous spectrum coincides with $[0, \infty)$, and its negative spectrum consists of eigenvalues.

Since the usual wave operators do not exist for $\rho \leq 1$, the large-time asymptotics of $e^{-itH}u$ for vectors u from the absolutely continuous subspace of H is described in terms of the modified free evolution. There are several possibilities to construct it. For example, in

coordinate representation the modified free evolution $U_0(t)$ is defined in [14] by the equation

$$(U_0(t)u)(x) = e^{i\Xi(x,t)} (2it)^{-d/2} \hat{u}(x/(2t)), \quad (2.1)$$

where

$$\Xi(x, t) = (4t)^{-1} |x|^2 - t \int_0^1 V(sx) ds, \quad (2.2)$$

and

$$\hat{u}(\xi) = (Fu)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x, \xi \rangle} u(x) dx$$

is the Fourier transform of u . Then the modified wave operators

$$W_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{itH} U_0(t) \quad (2.3)$$

exist and have the intertwining property $HW_{\pm} = W_{\pm}H_0$. Moreover, they are asymptotically complete, i.e., their ranges coincide with the absolutely continuous subspace of H .

Equivalently, the modified free dynamics can be defined (see [3] and [2]) in momentum representation by the equation

$$(F\tilde{U}_0(t)u)(\xi) = e^{-i|\xi|^2 t} e^{-i \int_0^t V(2\xi s) ds} \hat{u}(\xi).$$

Although the operators $U_0(t)$ and $\tilde{U}_0(t)$ do not coincide, W_{\pm} equals the wave operator

$$\tilde{W}_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{itH} \tilde{U}_0(t).$$

Still another possibility is to define the modified free dynamics by the introduction of an appropriate time-independent modifier [6, 7]. In this case the role of $U_0(t)$ in the definition of W_{\pm} is played by the operator $J_{\pm} e^{-itH_0}$ where J_{\pm} is a specially constructed pseudodifferential operator.

Given the wave operators, the scattering operator and the scattering matrix are defined exactly as in the short-range case. It follows from properties of the wave operators that the scattering operator

$$\mathbf{S} = W_+^* W_- \quad (2.4)$$

commutes with H_0 and is unitary in the space $L^2(\mathbb{R}^d)$. Let \mathbb{S}^{d-1} be the unit sphere in \mathbb{R}^d , $\mathbb{R}^+ = (0, \infty)$ and let $L^2(\mathbb{R}^+, L^2(\mathbb{S}^{d-1}))$ be the L^2 -space of functions defined on \mathbb{R}^+ with values in $L^2(\mathbb{S}^{d-1})$. Define the unitary operator

$$\mathcal{F} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^+, L^2(\mathbb{S}^{d-1}))$$

by the equation

$$(\mathcal{F}u)(\omega; \lambda) = 2^{-1/2} \lambda^{(d-2)/4} \hat{u}(\lambda^{1/2} \omega).$$

The spectral parameter λ plays the role of the energy of a quantum particle. Then $(\mathcal{F}H_0u)(\lambda) = \lambda(\mathcal{F}u)(\lambda)$ and

$$(\mathcal{F}\mathbf{S}u)(\lambda) = S(\lambda)(\mathcal{F}u)(\lambda).$$

The unitary operator $S(\lambda) : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ is known as the scattering matrix at energy λ .

Of course, the definition of the modified free dynamics and hence of modified wave operators is not unique. However, the freedom in their choice is rather limited. For example, in definition (2.1) one can add to $\Xi(x, t)$ an arbitrary (smooth) function which behaves as $\theta_{\pm}(x/(2t))$ for $t \rightarrow \pm\infty$. Then the wave operator (2.3) is replaced by $W_{\pm}e^{i\theta_{\pm}(\mathbf{p})}$ where $\mathbf{p} = -i\nabla$ and $e^{i\theta_{\pm}(\mathbf{p})} = F^*e^{i\theta_{\pm}(\xi)}F$ is multiplication by $e^{i\theta_{\pm}(\xi)}$ in the momentum representation. It follows that the scattering operator (2.4) is replaced by $\tilde{\mathbf{S}} = e^{-i\theta_+(\mathbf{p})}\mathbf{S}e^{i\theta_-(\mathbf{p})}$ and the scattering matrix $S(\lambda)$ is replaced by

$$\tilde{S}(\lambda) = e^{-i\theta_+(\sqrt{\lambda} \cdot)} S(\lambda) e^{i\theta_-(\sqrt{\lambda} \cdot)}. \quad (2.5)$$

In particular, if V is a sum of long-range $V_{lr}(x)$ and short-range $V_{sr}(x)$ functions, then V can be replaced by $V_{lr}(x)$ in (2.2). In this case

$$\theta_{\pm}(\xi) = 2^{-1} \int_0^{\pm\infty} V_{sr}(\xi s) ds.$$

We accept below that the scattering matrix is defined in terms of wave operators (2.3) with phase function (2.2).

3 The Structure of the Scattering Matrix

We need to know only the leading singularity of the scattering amplitude. The following result was essentially obtained in [15], but it is also a consequence of more general results of [18] where a complete description of all singularities was found.

THEOREM 3.1. *Suppose that estimate (1.1) holds for all α . Then the scattering matrix $S(\lambda)$ is a pseudodifferential operator on the unit sphere \mathbb{S}^{d-1} with the symbol*

$$a(y, \omega; \lambda) = e^{-i(2k)^{-1}\Phi(y, \omega)} (1 + b(y, \omega; \lambda)), \quad k = \lambda^{1/2}, \quad (3.1)$$

where Φ is function (1.6) and $b \in \mathcal{S}_{1,0}^{-2\rho+1}$ if $\rho < 1$ and $b \in \mathcal{S}_{1,0}^{-1+\varepsilon}$ for any $\varepsilon > 0$ if $\rho = 1$.

REMARK 3.2. Scattering matrix (2.5) is also a pseudodifferential operator with the symbol

$$\tilde{a}(y, \omega; \lambda) = e^{-i(2k)^{-1}\tilde{\Phi}(y, \omega)} (1 + \tilde{b}(y, \omega; \lambda)), \quad (3.2)$$

where

$$\tilde{\Phi}(y, \omega) = \Phi(y, \omega) + 2k\theta_+(k\omega) - 2k\theta_-(k\omega) \quad (3.3)$$

and \tilde{b} belongs to the same class as the function b in Theorem 3.1.

4 Inversion of the Radon transform

To solve our inverse scattering problem we use the two-dimensional Radon transform (see, e.g., [5]). Here we briefly recall some of its properties. For $v \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 1$, the Radon transform, or X -ray transform, which is the same in two dimensions, is defined by the formula

$$r(y, \omega; v) = \int_{-\infty}^{\infty} v(\omega t + y) dt, \quad \omega \in \mathbb{S}, \quad \langle \omega, y \rangle = 0.$$

It is clear that, $r(y, \omega) = r(y, -\omega)$. The Fourier transform \hat{v} of v and hence the function v itself can be reconstructed from its Radon transform in the following way. Let ω_ξ be one of the two unit vectors such that $\langle \omega_\xi, \xi \rangle = 0$. Hence,

$$\hat{v}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i|\xi|s} r(s\hat{\xi}, \omega_\xi; v) ds, \quad \hat{\xi} = \xi|\xi|^{-1}. \quad (4.1)$$

Below we apply this method for the reconstruction of a homogeneous function $V \in C^\infty(\mathbb{R}^d \setminus \{0\})$ of order $-\rho < 0$ from the integral $\Phi(y, \omega)$ defined by formula (1.6). We suppose that it is known for all $\omega \in \mathbb{S}^{d-1}$ and all $y \in \Pi_\omega$, $y \neq 0$. Actually, it suffices to know the function $\nabla \Phi(y, \omega)$ where $\nabla = \nabla_y$. For an arbitrary $x \in \mathbb{R}^d \setminus \{0\}$, we shall find $V(x)$.

Let us fix the coordinate system in such a way that the first axis is directed along x , i.e., $x = (x_1, 0, \dots, 0)$, and consider some two-dimensional plane Λ_x orthogonal to x . Suppose that $\omega \in \Lambda_x$, $|\omega| = 1$. Differentiating (1.6) with respect to y_1 , we find that

$$\partial_{y_1} \Phi(y, \omega; V) = \int_{-\infty}^{\infty} \partial_{y_1} V(y + t\omega) dt.$$

For $\bar{y} \in \Lambda_x$ such that $\langle \bar{y}, \omega \rangle = 0$, set

$$v_x(\bar{y}) := \partial_{x_1} V(x + \bar{y}).$$

Then, for all $\omega \in \Lambda_x$, $|\omega| = 1$, and all $\bar{y} \in \Lambda_x$, $\langle \bar{y}, \omega \rangle = 0$,

$$r(\bar{y}, \omega; v_x) = \partial_{y_1} \Phi(x + \bar{y}, \omega; V).$$

Since $x + \bar{y} \neq 0$, the function $v_x \in \mathcal{S}^{-\rho-1}(\Lambda_x)$ so that we can recover v_x and, in particular, $v_x(0) = \partial_{x_1} V(x)$ by formula (4.1). Then, integrating we reconstruct

$$V(x) = - \int_{x_1}^{\infty} \partial_s V(s, 0, \dots, 0) ds.$$

In particular, we have proven the following proposition.

PROPOSITION 4.1. *Let $V \in C^\infty(\mathbb{R}^d \setminus \{0\})$, $d \geq 3$, be a homogeneous function of order $-\rho < 0$. If $\nabla \Phi(y, \omega) = 0$ for all $\omega \in \mathbb{S}^{d-1}$ and all $y \in \Pi_\omega$, $y \neq 0$, then $V(x) = 0$.*

5 Reconstruction theorem

Reconstruction of the long-range part of the potential requires only the knowledge of the leading term in the asymptotics of the symbol $a(y, \omega; \lambda)$ as $|y| \rightarrow \infty$. The necessary result is formulated in Theorem 3.1.

First, we reconstruct the symbol $a(y, \omega; \lambda)$ from a family of the operators $S_{\omega_0}(\lambda)$ defined by equation (1.7). Here we use the fact that $a(y, \omega; \lambda)$ coincides with the symbol of the pseudodifferential operator $S_{\omega_0}(\lambda)$ for $\omega \in O_{\omega_0}$.

Then, by equation (3.1),

$$2ka(y, \omega; \lambda)^{-1} \nabla a(y, \omega; \lambda) = -i \nabla \Phi(y, \omega) + 2k(1 + b(y, \omega; \lambda))^{-1} \nabla b(y, \omega; \lambda). \quad (5.1)$$

Under condition (1.2), $\nabla b(1+b)^{-1} \in \mathcal{S}^{-p}$ where $p = 2\rho_1 > 1$ if $\rho_1 < 1$ and p is any number smaller than 2 if $\rho_1 = 1$, whereas

$$\nabla\Phi(y, \omega; V) = \sum_{j=1}^N \nabla\Phi(y, \omega; V_j) + \nabla\Phi(y, \omega; V_{sr}). \quad (5.2)$$

Here $\nabla\Phi(y, \omega; V_j)$, $j = 1, \dots, N$, are homogeneous functions of orders $-\rho_j \geq -1$ and $\nabla\Phi(\cdot, \omega; V_{sr}) \in \mathcal{S}^{-\rho_{sr}}$ with $-\rho_{sr} < -1$. Thus, given $a(y, \omega; \lambda)$, we single out in expression (5.1) all homogeneous terms of orders ≥ -1 . This yields us the functions $\nabla\Phi(y, \omega; V_j)$. Finally, each one of the V_j , $j = 1, \dots, N$, is uniquely reconstructed from $\nabla\Phi(y, \omega; V_j)$, as explained in the previous section.

Note that the number, N , of long-range terms, as well as their order of homogeneity, $-\rho_j$, $j = 1, \dots, N$, are obtained in the reconstruction process. We do not need to know them *a priori*.

In this way we have proven our main result.

THEOREM 5.1. *Suppose that $V \in C^\infty(\mathbb{R}^d)$, $d \geq 3$, and that for sufficiently large $|x|$ equality (1.2) is true. Here $V_j \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a homogeneous function of order $-\rho_j$ with $1/2 < \rho_1 < \rho_2 < \dots < \rho_N \leq 1$, and $V_{sr} \in C^\infty(\mathbb{R}^d)$ is a short-range potential that satisfies (1.1) for some $\rho_{sr} > 1$. Then, for an arbitrary $\lambda > 0$, any family of the operators $S_{\omega_0}(\lambda)$ uniquely determines each V_j , $j = 1, \dots, N$. Moreover, the functions V_j can be reconstructed from formulae (5.1), (5.2) by the inversion of the X-ray transform.*

REMARK 5.2. Let $\tilde{S}(\lambda)$ be the scattering matrix defined by formula (2.5). Considered as a pseudodifferential operator it has symbol (3.2). According to (3.3)

$$\nabla\tilde{\Phi}(y, \omega) = \nabla\Phi(y, \omega)$$

so that in view of (5.2) the functions V_j , $j = 1, \dots, N$, can be reconstructed from the function $\tilde{\Phi}$. Hence, we can reconstruct the long-range part of the potential from any one of the possible choices of scattering matrices, that correspond to different modified free dynamics.

References

- [1] Agmon S 1978 Some new results in spectral and scattering theory of differential operators in \mathbb{R}^n , *Seminaire Goulaouic-Schwartz* (France: Ecole Polytechnique)
- [2] Buslaev V and Matveev V 1970 Wave operators for the Schrödinger equation with a slowly decreasing potential, *Theor. Math. Phys.* **2** 266-274
- [3] Dollard J 1964 Asymptotic convergence and Coulomb interaction, *J. Math. Phys.* **5** 723-738.
- [4] Enss V and Weder R 1995 The geometrical approach to multidimensional inverse scattering, *J. Math. Phys.* **36** 3902-3921
- [5] Helgason S 1984 *Groups and Geometric Analysis* (New York: Academic Press)
- [6] Isozaki H and Kitada H 1985 Modified wave operators with time-independent modifiers, *J. Fac. Sci, Univ. Tokyo* **32** 77-104.
- [7] Isozaki H and Kitada H 1986 Scattering matrices for two-body Schrödinger operators, *Sci. Papers College Arts Sci. Univ. Tokyo* **35** 81-107
- [8] Joshi M S 1999 Recovering asymptotics of Coulomb-like potentials from fixed energy scattering data, *SIAM J. Math. Anal.* **30** 516-526
- [9] Nicoleau F 2004 A constructive procedure to recover asymptotics of short-range potentials, *J. Diff. Eq.* **205** 354-364
- [10] Roux Ph and Yafaev D The scattering matrix for the Schrödinger operator with a long-range electromagnetic potential, *J. Math. Phys.* **44** (2003), 2762-2786
- [11] Shubin M A 1987 *Pseudodifferential Operators and Spectral Theory* (Berlin: Springer-Verlag)
- [12] Treves F 1980 *Introduction to Pseudodifferential and Fourier Integral Operators Volume 1 Pseudodifferential Operators, Volume 2 Fourier Integral Operators* (New York: Plenum Press)

- [13] Weder R and Yafaev D 2005 On inverse scattering at a fixed energy for potentials with a regular behaviour at infinity, *Inverse Problems* **21** 1937–1952
- [14] Yafaev D 1981 Wave operators for the Schrödinger equation, *Theoret. and Math. Phys.* **45** , 992-998.
- [15] Yafaev D 1998 The scattering amplitude for the Schrödinger equation with long-range potential, *Comm. Math. Phys.* **112** 245-255
- [16] Yafaev D 2000 A class of pseudodifferential operators with oscillating symbol, *St. Petersburg Math. J.* **11** 218-256
- [17] Yafaev D 2000 *Scattering Theory: Some Old and New Problems*. Lecture Notes in Math. **1735** (Berlin: Springer)
- [18] Yafaev D 2003 High-energy and smoothness asymptotic expansion of the scattering amplitude, *J. Funct. Anal.* **202** 526-570